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Laser Ray Trace Approach
To Cross-Beam Energy Transfer
in HYDRA

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Abstract

We estimate the laser intensity in each zone by summing contributions from each ray, resolving the sums per beam for a set of beams. These zonal beam intensities are used to compute the cross-beam energy transfer between each beam and the ray as it traverses the zone. The ray trace computation is iterated until the intensities converge.

Contents

1 Power Deposition By Inverse Bremsstrahlung

Because rays are simply curves in space, they carry no information about radiation intensity or spatial extent transverse to their direction. Their state is completely defined by their frequency, velocity and power, the latter two attributes of which are, in general, spatially dependent. The power of an electromagnetic wave is depleted as the oscillatory energy it imparts to electrons is randomized by collisions, the inverse-bremsstrahlung process. The rate of energy loss is given by well-known formula

$$\nu_{ib} = \frac{n_e}{n_c}\, \nu_{ei}, \tag{1}$$

where

$$\nu_{ei} = \frac{4}{3} \left(\frac{2\pi}{m_e}\right)^{1/2} \frac{n_e Z e^4 \ln \Lambda}{T_e^{3/2}} \tag{2}$$

is the electron-ion collision rate. As a ray traverses a zone its power decreases with time:

$$P(\Delta t) = P(0) \exp\{-\int_0^{\Delta t} dt\, \nu_{ib}(\vec{x}(t))\} \tag{3}$$

Δt is the time required to traverse the zone, $\vec{x}(t)$ is given by

$$\vec{v}(\Delta t) = \vec{v}_0 - \frac{c^2}{2n_c}\langle \vec{\nabla} n_e \rangle\, \Delta t, \tag{4}$$

$$\vec{x}(\Delta t) = \vec{x}_0 + \vec{v}_0\, \Delta t - \frac{c^2}{4n_c}\langle \vec{\nabla} n_e \rangle (\Delta t)^2, \tag{5}$$

and $\vec{x}_0$, $\vec{v}_0$ are the entry position and velocity. Note:

$$|\vec{v}_0| = v_g = c\eta = c\sqrt{1 - n_e/n_c}. \tag{6}$$

Because ν_{ib} can be a strongly non-uniform function of position within a zone, care must be taken in computing the integral in (3). With the approximation

$$n_e(\vec{x}) = \langle n_e \rangle + \langle \vec{\nabla} n_e \rangle \cdot (\vec{x} - \langle \vec{x} \rangle) + O(\epsilon^2) \tag{7}$$

where $\langle\cdot\rangle$ denotes a zone average for $n_e(\vec{x}(t))$ and similar linear approximations for T_e, and the gradients $\langle\vec{\nabla}n_e\rangle$ and $\langle\vec{\nabla}T_e\rangle$, the integrand becomes

$$\nu_{ib}(\vec{x}(t)) = \nu_0 \frac{(1 + Ut + Rt^2)^2}{(1 + Wt + St^2)^{3/2}}, \qquad (8)$$

where

$$
\begin{aligned}
U &= \frac{\vec{v}_0 \cdot \langle\vec{\nabla}n_e\rangle}{n_{e0}}, \\
W &= \frac{\vec{v}_0 \cdot \langle\vec{\nabla}T_e\rangle}{T_{e0}}, \\
R &= -\frac{c^2}{4} \frac{\langle\vec{\nabla}n_e\rangle \cdot \langle\vec{\nabla}n_e\rangle}{n_c n_{e0}}, \\
S &= -\frac{c^2}{4} \frac{\langle\vec{\nabla}n_e\rangle \cdot \langle\vec{\nabla}T_e\rangle}{n_c T_{e0}}.
\end{aligned}
\qquad (9)
$$

The quantities

$$
\begin{aligned}
\nu_0 &\equiv \frac{4}{3}\left(\frac{2\pi}{m_e}\right)^{1/2} \frac{Ze^4}{n_c} \frac{n_{e0}^2 \ln\Lambda_0}{T_{e0}^{3/2}}, & (10) \\
n_{e0} &\equiv \langle n_e\rangle + \langle\vec{\nabla}n_e\rangle \cdot (\vec{x}_0 - \langle\vec{x}\rangle), & (11) \\
T_{e0} &\equiv \langle T_e\rangle + \langle\vec{\nabla}T_e\rangle \cdot (\vec{x}_0 - \langle\vec{x}\rangle), & (12)
\end{aligned}
$$

and $\ln\Lambda_0$ are defined at the zone entry point. The integral can be evaluated in closed form, but the result is not computationally simple enough to be useful, nor warranted on accuracy grounds. A more efficient approach with sufficient accuracy is Gaussian quadrature:

$$\int_0^{\Delta t} dt\, \nu_{ib}(\vec{x}(t)) = \nu_0 \frac{\Delta t}{2} \sum_{i=1}^{N_g} w_i \frac{(1 + Ut_i + Rt_i^2)^2}{(1 + Wt_i + St_i^2)^{3/2}}, \qquad (13)$$

where N_g is the order of the integration scheme, and w_i the i-th Gaussian weight. The evaluation times are given by $t_i = (\xi_i + 1)\Delta t/2$, with ξ_i the i-th Gaussian abscissa.

The rate at which energy is deposited in the electrons in the zone is $P(0) - P(\Delta t)$, which can be used as a source term in an electron energy equation.

2   Laser Field Intensity

In addition to providing a source for electron energy transport, the laser-plasma interaction model can also furnish a momentum source by way of ponderomotive effects, which depend on the laser-field energy density and its gradient. The contribution of a single ray to the intensity of the laser field in a zone can be computed as $\mathcal{I}_r = \overline{P}\tau_r/\Delta V$, where $\tau_r = (\int cds/v_g)_r = (\int ds/\eta)_r$ is the time it takes the ray to cross the zone, ΔV is the zone volume and $\overline{P}$ is the time-averaged power

$$\overline{P} = \frac{1}{\tau}\int_0^\tau d\tau' P(\tau') \tag{14}$$

with $P(\tau)$ given by (3),(13). Note:τ has length units since it is time scaled by c. In HYDRA, we use the Gaussian quadrature result to obtain a cell average $\overline{\nu}_{ib}$ for each ray, then evaluate (14) directly as

$$\overline{P} = \frac{1}{\tau}\int_0^\tau d\tau' P(\tau') = P(0)\frac{1 - \exp(-\overline{\nu}_{ib}\tau)}{\overline{\nu}_{ib}\tau} \tag{15}$$

so that for each ray

$$\mathcal{I}_r = \overline{P}\tau_r/\Delta V = P(0)\frac{1 - \exp(-\overline{\nu}_{ib}\tau)}{\overline{\nu}_{ib}\Delta V}. \tag{16}$$

For small values of $\overline{\nu}_{ib}\tau_r$, $\mathcal{I}_r$ reduces to

$$\lim_{\overline{\nu}_{ib}\tau_r \to 0} \mathcal{I}_r = P(0)(1 - \frac{\overline{\nu}_{ib}\tau_r}{2})\frac{\tau_r}{\Delta V} \tag{17}$$

The total intensity for all rays traversing the cell is then just the sum

$$\mathcal{I} = \sum_r \mathcal{I}_r. \tag{18}$$

3   Cross Beam Coupling

Michel's equations (6) and (7) are

$$\frac{\partial}{\partial z}|a_1|^2 = 2\Re(\frac{-i\Delta k^2 F_{\chi,1}}{8k_1})|a_1|^2|a_2|^2 \tag{19}$$

$$\frac{\partial}{\partial z}|a_2|^2 = 2\Re(\frac{-i\Delta k^2 F_{\chi,2}}{8k_2})|a_2|^2|a_1|^2 \tag{20}$$

with $\Delta k^2 = |\vec{k_1} - \vec{k_2}|^2$, $\vec{k_1} = \eta_1 \vec{k_1^v}$, $\vec{k_2} = \eta_2 \vec{k_2^v}$ and

$$F_{\chi,1} = \frac{\chi_e(1+\chi_i)}{1+\chi_e+\chi_i},_1, \tag{21}$$

and $\vec{k^v}$ are the beam vacuum wave-numbers and $\chi_e(\Delta k, \Delta\omega, n_e, T_e)$ and $\chi_i(\Delta k, \Delta\omega, n_i, T_i)$ are the electron and ion susceptibilites which we'll come to presently.

Writing

$$2\Re(\frac{-i\Delta k^2 F_{\chi,1}}{8k_1}) = \frac{\Delta k^2}{4k_1}\Im(\frac{\chi_e(1+\chi_i)}{1+\chi_e+\chi_i}),_1 \tag{22}$$

$$= \frac{\Delta k^2}{4k_1}\frac{\Im(\chi_e)(1+2\Re(\chi_i)+|\chi_i|^2)+\Im(\chi_i)|\chi_e|^2}{|1+\chi_e+\chi_i|^2},_1 \tag{23}$$

$$= \frac{\Delta k^2}{4k_1}\frac{\Im(\chi_e)(1+2\Re(\chi_i)+|\chi_i|^2)+\Im(\chi_i)|\chi_e|^2}{1+|\chi_e|^2+|\chi_i|^2+2\Re(\chi_e+\chi_i+\chi_e\chi_i^\star)},_1 \tag{24}$$

$$= \mathcal{C}_{12} \tag{25}$$

we can cast (19), (20) in the form

$$\frac{\partial}{\partial z}|a_1|^2 = \mathcal{C}_{12}|a_1|^2|a_2|^2 \tag{26}$$

$$\frac{\partial}{\partial z}|a_2|^2 = \mathcal{C}_{21}|a_2|^2|a_1|^2. \tag{27}$$

In terms of the beam intensities $\alpha_{1,2}^{-1}|a_{1,2}|^2 \equiv \mathcal{I}_{1,2}$ with

$$\alpha_{1,2}^{-1} = \frac{\epsilon_0}{2}(\frac{(m_ec^2)}{e})^2 k_{1,2}^v \omega_{1,2}$$

$$\frac{\partial}{\partial z}\alpha_1\mathcal{I}_1 = \mathcal{C}_{12}\alpha_1\mathcal{I}_1\alpha_2\mathcal{I}_2 \tag{28}$$

$$\frac{\partial}{\partial z}\alpha_2\mathcal{I}_2 = \mathcal{C}_{21}\alpha_2\mathcal{I}_2\alpha_1\mathcal{I}_1. \tag{29}$$

Alternatively, with $v_g = c\eta = c^2 k/\omega$ and $ds = \eta d\tau$ to measure path length along rays

$$\frac{\partial}{\partial \tau_1}\mathcal{I}_1 = \eta_1\alpha_2\mathcal{C}_{12}\,\mathcal{I}_2\mathcal{I}_1 \tag{30}$$

$$\frac{\partial}{\partial \tau_2}\mathcal{I}_2 = \eta_2\alpha_1\mathcal{C}_{21}\,\mathcal{I}_1\mathcal{I}_2. \tag{31}$$

Accounting for uncorrelated polarizations modifies (30),(31) slightly to give

$$\frac{\partial}{\partial \tau_1}\mathcal{I}_1 = p_{12}\eta_1\alpha_2\mathcal{C}_{12}\,\mathcal{I}_2\mathcal{I}_1 \tag{32}$$

$$\frac{\partial}{\partial \tau_2}\mathcal{I}_2 = p_{21}\eta_2\alpha_1\mathcal{C}_{21}\,\mathcal{I}_1\mathcal{I}_2 \tag{33}$$

where

$$p_{12} = p_{21} = \frac{1}{4}(1 + \cos^2\theta_{12})$$

and θ_{12} is the angle between the beam wave-vectors

$$\cos\theta_{12} = \frac{\vec{k_1}\cdot\vec{k_2}}{|\vec{k_1}||\vec{k_2}|}.$$

3.1   Evaluation of Coupling Coefficients

The coupling coefficients $\mathcal{C}_{12}, \mathcal{C}_{21}$ require evaluation of the electron and ion susceptibilites χ_e, χ_i with respect to the beat wave ion acoustic wave which couples the two beams and mediates the energy transfer. χ_e, χ_i are given by

$$\chi_{e,1} = \frac{-1}{2(k\lambda_{De})^2} Z' \left(\frac{\omega_2 - \omega_1 - (\vec{k}_2 - \vec{k}_1) \cdot \vec{V}}{\sqrt{2}\omega_{pe}k\lambda_{De}} \right) \tag{34}$$

$$\equiv \alpha_e Z'(\beta_e), \tag{35}$$

$$\chi_{i,1} = \sum_s \frac{-1}{2(k\lambda_{Di_s})^2} Z' \left(\frac{\omega_2 - \omega_1 - (\vec{k}_2 - \vec{k}_1) \cdot \vec{V}}{\sqrt{2}\omega_{pi_s}k\lambda_{Di_s}} \right) \tag{36}$$

$$= \sum_s \left(\frac{n_{i_s} T_e Z_{i_s}^2}{n_e T_{i_s}} \right) \frac{-1}{2(k\lambda_{De})^2} Z' \left(\sqrt{\frac{\mu_{i_s} T_e}{T_{i_s}}} \frac{\omega_2 - \omega_1 - (\vec{k}_2 - \vec{k}_1) \cdot \vec{V}}{\sqrt{2}\omega_{pe}k\lambda_{De}} \right) \tag{37}$$

$$\equiv \sum_s \left(\frac{n_{i_s} T_e Z_{i_s}^2}{n_e T_{i_s}} \right) \alpha_e Z'(\sqrt{\frac{\mu_{i_s} T_e}{T_{i_s}}} \beta_e). \tag{38}$$

with $k^2 = |\vec{k}_2 - \vec{k}_1|$ throughout, $\mu_{i_s} = m_{i_s}/m_e$ and $Z'(x)$ for real x is

$$Z'(x) = -2 \left(1 + xe^{-x^2} \left(i\sqrt{\pi} - 2 \int_0^x dt \, e^{t^2} \right) \right). \tag{39}$$

Using the definition for Z_{eff}

$$Z_{\text{eff}} \equiv \frac{\sum_s n_{i_s} Z_{i_s}^2}{\sum_s n_{i_s} Z_{i_s}} = \frac{\sum_s n_{i_s} Z_{i_s}^2}{n_e}$$

a possible approximation for $\chi_{i,1}$ when $T_{i_s} \sim T_i$ and μ_{i_s} is suitably averaged is given by the expression

$$\chi_{i,1} = \sum_s \left(\frac{n_{i_s} T_e Z_{i_s}^2}{n_e T_{i_s}} \right) \alpha_e Z'(\sqrt{\frac{\mu_{i_s} T_e}{T_{i_s}}} \beta_e) \tag{40}$$

$$\sim Z_{\text{eff}} \frac{T_e}{T_i} \alpha_e Z'(\sqrt{\frac{\mu_i T_e}{T_i}} \beta_e). \tag{41}$$

$\Re(Z'(x))$ is even with respect to x while $\Im(Z'(x))$ is odd;

$$\Re(Z'(x)) = \Re(Z'(-x)) \tag{42}$$

$$\Im(Z'(x)) = -\Im(Z'(-x)). \tag{43}$$

The expansion for $Z'(x)$ for small x is

$$\Re(Z'(x)) = -2(1 - 2xD(x)) \tag{44}$$

$$\Im(Z'(x)) = -2\sqrt{\pi}xe^{-x^2}. \tag{45}$$

with the Dawson Function $D(x) = -D(-x)$

$$D(x) = x - \frac{2}{3}x^3 + \frac{2^2}{5 \cdot 3}x^5 - \frac{2^3}{7 \cdot 5 \cdot 3}x^7 + \dots \tag{46}$$

$$= \sum_{n=0}^{\infty}(-)^n\frac{2^{2n}n!}{(2n+1)!}x^{2n+1} \tag{47}$$

$$= \sum_{n=1}^{\infty} D_{2n-1}x^{2n-1} \tag{48}$$

and

$$D_{2n-1} = D_{2(n-1)-1}\frac{-2}{2(n-1)-1}.$$

For large argument the expansion for $Z'(x)$ is

$$\Re(Z'(x)) \sim \frac{1}{x^2} + \frac{3}{2x^4} + \frac{3 \cdot 5}{4x^6} + \dots \tag{49}$$

$$\Im(Z'(x)) \sim 0 \tag{50}$$

A fast algorithm for computing the Dawson Function $D(x)$ can be found in the appendix.

3.2   Integration Into Ray Trace Algorithm

Representing (84), (85) in terms of the ray powers using as before $\mathcal{I}_r = \overline{P}_r\tau_r/\Delta V$, where $\tau_r = (\int ds\, c/v_g)_r = (\int ds/\eta)_r$ for each ray, we write for rays (r_1, r_2) in beams (b_1, b_2)

$$\frac{\partial}{\partial\tau_1}\sum_{r_1}\overline{P}_{r_1}\tau_{r_1}|_2 = \alpha_2\eta_1\mathrm{p}_{12}\mathcal{C}_{12}\sum_{r_1}\overline{P}_{r_1}\tau_{r_1}\sum_{r_2}\overline{P}_{r_2}\tau_{r_2}/\Delta V \tag{51}$$

$$\frac{\partial}{\partial\tau_2}\sum_{r_2}\overline{P}_{r_2}\tau_{r_2}|_1 = \alpha_1\eta_2\mathrm{p}_{12}\mathcal{C}_{21}\sum_{r_1}\overline{P}_{r_1}\tau_{r_1}\sum_{r_2}\overline{P}_{r_2}\tau_{r_2}/\Delta V \tag{52}$$

noting here that the beam intensities in each of these equations represent a sum over all rays of that beam transiting through the zone. For more than two beams, the more general form of (51), for example, would be

$$\frac{\partial}{\partial \tau_1} \sum_{r_1} \overline{P}_{r_1} \tau_{r_1} = \sum_q \left(\alpha_q \eta_1 \mathrm{p}_{1q} \mathcal{C}_{1q} \sum_{r_1} \overline{P}_{r_1} \tau_{r_1} \sum_{r_q} \overline{P}_{r_q} \tau_{r_q} \right) / \Delta V, \qquad (53)$$

where the sum over q includes all beams with intensities in the zone. Integrating (51),(51) along a single ray in the zone we get the power transfer relations

$$\delta \overline{P}_{r_1}|_2 = \alpha_2 \eta_1 \mathrm{p}_{12} \mathcal{C}_{12} \, \overline{P}_{r_1} \tau_{r_1} \mathcal{I}_2 \qquad (54)$$

$$\delta \overline{P}_{r_2}|_1 = \alpha_1 \eta_2 \mathrm{p}_{12} \mathcal{C}_{21} \, \overline{P}_{r_2} \tau_{r_2} \mathcal{I}_1. \qquad (55)$$

We note that due to (42),(43) the cross-beam power transfers for beams (b_1, b_2) from (54), (55) are related by

$$\frac{\sum_{r_1} \delta \overline{P}_{r_1}|_2}{\sum_{r_2} \delta \overline{P}_{r_2}|_1} = \frac{\alpha_2 \eta_1 \mathcal{C}_{12}}{\alpha_1 \eta_2 \mathcal{C}_{21}} = -\frac{\alpha_2 \eta_1 k_2}{\alpha_1 \eta_2 k_1} = -\frac{\lambda_2^v}{\lambda_1^v} = -\frac{\omega_1}{\omega_2} \qquad (56)$$

leading, after summing over the rays traversing the zone to the conservation relation

$$\Delta P_{12} \lambda_1^v = -\Delta P_{21} \lambda_2^v. \qquad (57)$$

From (57) we can represent the power transferred to the ion acoustic wave to be

$$\Delta P^{ia} = \Delta P_{12} + \Delta P_{21} = \Delta P_{12} \left(1 - \frac{\omega_2 - (\vec{k}_2 - \vec{k}_1) \cdot \vec{V}}{\omega_1} \right). \qquad (58)$$

We include the Doppler shift to account for heat flow to the plasma through the ion acoustic wave damping. The momentum transfer due to cross beam transfer is accounted for implicitly in the ray trace algorithm through the ponderomotive terms. In this way directed energy added to the plasma through the ponderomotive forces and heat flow due to damping of the ion acoustic ewaves should enter the total energy balance correctly.

It is algorithmically convenient to symmetrize (58) per ray to simplify the accumulation over rays of the total power transferred to ion acoustic waves due the cross-beam power transfers for beams (b_1, b_2). We do this as follows:

$$\delta \overline{P}_{r_1}^{ia}|_2 = \frac{1}{2} \delta \overline{P}_{r_1}|_2 \left(1 - \frac{\omega_2 - (\vec{k}_2 - \vec{k}_1) \cdot \vec{V}}{\omega_1} \right) \tag{59}$$

$$\delta \overline{P}_{r_2}^{ia}|_1 = \frac{1}{2} \delta \overline{P}_{r_2}|_1 \left(1 - \frac{\omega_1 - (\vec{k}_1 - \vec{k}_2) \cdot \vec{V}}{\omega_2} \right). \tag{60}$$

4   Intensity Iteration Algorithm

Solving equations (53) self-consistently in Hydra's LZR module requires an iteration scheme since the intensities and the ray powers are interdependent. The rays must be traced each cycle to determine the zonal intensities. On the nth iteration, we can adjust the zonal ray power depletion due to inverse Bremsstrahlung to include the cross beam coupling by

$$\nu_{r_1}^n \to \nu_{ibr_1} - \sum_q \alpha_q \mathrm{p}_{1q} \eta_1 \mathcal{C}_{1q} \mathcal{I}_q^{\,n-1} = \nu_{ibr_1} - \sum_q \nu_{1q}^{n-1}. \tag{61}$$

The intensities $\mathcal{I}_q^{\,n-1}$ here are inaccurate since they have been computed in the previous ray tracing iteration. The adjustment must be iterated to convergence. In each iteration we adjust the depletion modified by the cross beam coupling along all ray paths as described in (61). The zonal intensities are recalculated during each iteration by accumulating the contributions for each ray in each beam as in (18), but using (61) to include the cross beam coupling. Thus, for example, the beam intensity in a zone due to beam b_q that we use for the nth iteration is the sum of the contributions to that intensity of all the beam b_q rays traversing that zone at iteration count $n-1$, viz.

$$\mathcal{I}_q^{\,n-1} = \sum_{r_q} \mathcal{I}_{q_{r_q}}^{\,n-1} = \sum_{r_q} \overline{P}_{r_q}^{\,n-1} \tau_{r_q} / \Delta V. \tag{62}$$

The total zonal ray power change is then given by

$$\Delta \overline{P}_{r_1}^{n-1} = \overline{P}_{r_1}^{n-1}(0)(1 - \exp\left(-\int_0^{\tau_{r_1}} d\tau \, \nu_{ibr_1} + \tau_{r_1} \sum_q \nu_{1q}^{n-1}\right). \tag{63}$$

Dropping the superscript $^{n-1}$ for ease of notation, we have the zonal ray power depletion due to inverse bremsstrahlung for ray r_1 is

$$\delta \overline{P}_{r_1}|_{ib} = \Delta \overline{P}_{r_1} \frac{-\overline{\nu}_{ibr_1}}{-\overline{\nu}_{ibr_1} + \sum_{q'} \nu_{1q'}} \tag{64}$$

with

$$\overline{\nu}_{ibr_1} = \frac{1}{\tau_{r_1}} \int_0^{\tau_{r_1}} d\tau \, \nu_{ibr_1}(\tau).$$

The zonal power transferred due to cross beam coupling of ray r_1 with beam b_q is

$$\delta \overline{P}_{r_1}|_q = \Delta \overline{P}_{r_1} \frac{\nu_{1q}}{-\frac{1}{\tau_{r_1}} \int_0^{\tau_{r_1}} d\tau \, \nu_{ibr_1} + \sum_{q'} \nu_{1q'}} \tag{65}$$

and the associated power transferred to the ion acoustic wave, by (58), is

$$\delta \overline{P}_{r_1}^{ia}|_q = \frac{1}{2} \delta \overline{P}_{r_1}|_q \left(1 - \frac{\omega_q - (\vec{k}_q - \vec{k}_1) \cdot \vec{V}}{\omega_1}\right) \tag{66}$$

$$= \frac{1}{2} \Delta \overline{P}_{r_1} \frac{\nu_{1q}}{-\frac{1}{\tau_{r_1}} \int_0^{\tau_{r_1}} d\tau \, \nu_{ibr_1} + \sum_{q'} \nu_{1q'}} \left(1 - \frac{\omega_q - (\vec{k}_q - \vec{k}_1) \cdot \vec{V}}{\omega_1}\right). \tag{67}$$

To get the total cross beam power transfer between beams $(b_q, b_{q'})$ we need only sum the contributions in (65) appropriately:

$$\Delta \overline{P}_{qq'} = \sum_{cells_{qq'}} \sum_{r_q} \delta \overline{P}_{r_q}|_{q'}. \tag{68}$$

In (68) the sum over cells includes all cells traversed by rays from both beam b_q and $b_{q'}$ simultaneously. Note that the r_q ray b_q coupling for any cell in the set $cells_{qq'}$ to beam $b_{q'}$ denoted by $\Delta \overline{P}_{qq'}$ in (68) includes all traversals

by all rays in beam $b_{q'}$ for that cell, as shown in (62); ie. all $r_{q'}$ rays in beam $b_{q'}$ contribute to the intensity $\mathcal{I}_{q'}$ for the coupling.

From energy conservation, our iteration scheme converges when

$$\Delta \overline{P}_{qq'} + \Delta \overline{P}_{q'q} = \Delta \overline{P}_{qq'}^{ia} = \Delta \overline{P}_{q'q}^{ia}. \tag{69}$$

This condition is a useful check to verify the computation.

4.1 Intensity Iteration Algorithm Acceleration

In the earliest implementations of the intensity iteration scheme the entire ray tracing computation was redone for each iteration of the intensities, re-computing the cross beam power transfer coefficients for each iteration for those cells illuminated by more than one beam group. A series of improvements have followed the initial development to streamline and accelerate the computation. In order of their implementation they are: (1) Compute ray initialization (ray entry calculation) only once per cycle; (2) Compute ray geometry and recompute the cross beam power transfer coefficients for a limited number of iterations, LZR_XBET_iter_lite $>= 0$, saving the essential geometric and coefficient data for further iteration, rescaling coefficients by updated intensities; (3) Iterate only until a prespecified energy conservation tolerance is met, within a maximum iteration limit; (4) Save and reset intensity state across cycles as a preconditioning scheme for subsequent iterations LZR_XBET_istate$(0, 1)$.

Each of these 4 improvements have been effective in reducing the computational load of the XBET calculation.

4.2 Local Wavevector LZR_XBET_klocal

Effects of refraction on cross beam power transfer are modeled more precisely by specifying the wavevector associated with beam group resolved intensities on a per cell basis. The earliest implementations took the wavevector direction to be a beam group constant. Setting parameter LZR_XBET_klocal$(0, 1)$ causes each iteration to accumulate an intensity weighted wavevector direction for each beam group in each cell. These quantities are used for any coefficient recomputation requested and they are saved and reset across cycles as part of the preconditioning scheme for subsequent iterations LZR_XBET_istate$(0, 1)$.

5 Saturation Model

Michel suggests a saturation model in which the plasma wave amplitude driven by each beam pair interaction is limited from above by the value $|\delta n/n_e| = \texttt{LZR\_XBET\_saturation} \sim 4 \times 10^{-4}$. The plasma wave amplitude associated with beams (q, q') by Michel's equation (33) is

$$\left|\frac{\delta \hat{n}_e}{n_e}\right|_{q,q'} = \sqrt{\mathrm{p}_{qq'}}\, |F_\chi|_{q,q'} \left(\frac{\Delta k^2 c^2}{2\omega_{pe}^2}\right) \sqrt{a_q \mathcal{I}_q\, \alpha_{q'} \mathcal{I}_{q'}} \tag{70}$$

$$\equiv \mathcal{C}_{qq'}^{n_e} \sqrt{\mathcal{I}_q \mathcal{I}_{q'}}. \tag{71}$$

Using (70) to restructure (54) we have the expression

$$\nu_{qq'} \equiv \frac{\partial \ln \delta \overline{P}_{r_q}|_{q'}}{\partial \tau} \tag{72}$$

$$= \alpha_{q'} \eta_q \mathrm{p}_{qq'} \mathcal{C}_{qq'} \mathcal{I}_{q'} \tag{73}$$

$$= \left|\frac{\delta \hat{n}_e}{n_e}\right|_{q,q'} Q_{q,q'} \tag{74}$$

$$= \mathcal{C}_{qq'}^{n_e} \sqrt{\mathcal{I}_q \mathcal{I}_{q'}}\, \mathcal{C}_{qq'}^Q \sqrt{\mathcal{I}_{q'}/\mathcal{I}_q} \tag{75}$$

where

$$\mathcal{C}_{qq'}^{n_e} = \sqrt{\mathrm{p}_{qq'} a_q \alpha_{q'}} \left(\frac{\Delta k^2 c^2}{2\omega_{pe}^2}\right) |F_\chi|_{q,q'} \tag{76}$$

and

$$\mathcal{C}_{qq'}^Q = \eta_q \sqrt{\mathrm{p}_{qq'} \frac{\alpha_{q'}}{a_q}} \left(\frac{\omega_{pe}^2}{2k_q c^2}\right) \frac{\Im(F_{\chi_{q,q'}})}{|F_\chi|_{q,q'}} \tag{77}$$

and all $\mathcal{I}_q$ are taken at the previous iteration, $\mathcal{I}_q{}^{n-1}$.
Since (70) is unchanged by $(q, q') \to (q', q)$ it is evident that

$$\left|\frac{\delta \hat{n}_e}{n_e}\right|_{q,q'} = \left|\frac{\delta \hat{n}_e}{n_e}\right|_{q',q}. \tag{78}$$

By a similar argument as that leading to (56) we have that

$$\frac{Q_{q,q'}}{Q_{q',q}} = -\frac{\omega_q \mathcal{I}_{q'}}{\omega_{q'} \mathcal{I}_q}. \tag{79}$$

We implement the saturation model by requiring that in the sum over q' in (73) each term observe the limit that $|\delta \hat{n}_e / n_e|_{q,q'}$ is the lesser of the RHS of (70) or a fixed saturation threshold $S_t \sim 4 \times 10^{-4}$. So (73) becomes

$$\nu_{qq'} = \left| \frac{\delta \hat{n}_e}{n_e} \right|_{q,q'} Q_{q,q'} = \mathcal{C}_{qq'}^{n_e} \sqrt{\mathcal{I}_q \mathcal{I}_{q'}} \, \mathcal{C}_{qq'}^{Q} \sqrt{\mathcal{I}_{q'}/\mathcal{I}_q} \tag{80}$$

$$= \mathcal{C}_{qq'}^{n_e} \mathcal{C}_{qq'}^{Q} \mathcal{I}_{q'} \qquad\qquad \Longleftrightarrow \mathcal{C}_{qq'}^{n_e} \sqrt{\mathcal{I}_q \mathcal{I}_{q'}} < S_t \quad (81)$$

$$= S_t \mathcal{C}_{qq'}^{Q} \sqrt{\mathcal{I}_{q'}/\mathcal{I}_q} \qquad\qquad \Longleftrightarrow \mathcal{C}_{qq'}^{n_e} \sqrt{\mathcal{I}_q \mathcal{I}_{q'}} >= S_t \tag{82}$$

The saturation model uses the same iteration algorithm described previously.

6 Quasi-2D Cross Beam Energy Transfer

Hydra supports a quasi-2D ray tracing model often used in concert with the `rotmeshik` command and $jx = 2$ which automatically sets the `bm_axisymmetry` parameter. In this model the constant azimuthal angle surfaces $j = 1, 2$ are considered symmetry planes and all rays are specularly reflected there. Beams can be specified in the same general way as in the 3D case, but for purposes of ray initialization, all 3D rays are rotated so they are launched from the mid-plane of the cylindrical wedge between the $j = 1, 2$ (planar) surfaces. As rays are traced, the azimuthal coordinate of the ray in the 3D space is computed, but not used directly. The 2D ray representation remains within the wedge. One can think of the 2D ray as the 3D ray folded at the $j = 1, 2$ boundaries. For the quasi-2D cross beam energy transfer computation, it is not sufficient solely to identify the beam group of the ray segments traversing a given cell for coupling as in the 3D calculation. The folding procedure used to map all ray segments into the 2D wedge seems to bring together in each cell rays which are rotated in azimuth in the 3D world and thus may not be interacting at all. Reconstructing the actual 3D rotation corresponds to accumulating the initial ray rotation into the wedge mid-plane and each of the rotations corresponding to each reflection at a symmetry boundary.

6.1 `LZR_XBET_extrude`

The situation is rectified by calculating the *coincidence* of the 3D azimuth of the ray with the 3D beam crossection of the beam group the ray may be interacting with in the cell in question. There are two methods used for this purpose selected for by parameter `LZR_XBET_extrude` (0,1). Historically the first method implemented, chosen by `LZR_XBET_extrude` (0), is approximate and costly. It estimates the overlap of beam groups pairwise from a heuristic geometric argument then applies this overlap factor on a per ray basis. `LZR_XBET_extrude` (1) is superior in virtually every way. In practice, the default `LZR_XBET_extrude` (1) is the preferred model. A brief description of method (1) follows; method (0) is described in the Appendix.

`LZR_XBET_extrude` (1) creates an azimuthally extruded pseudo-mesh to track the 3D rays and compute their cross-beam energy transfer interaction. The results are identical to those for the full 3D calculation corresponding to an exactly axisymmetric version of the 2D problem generated by rotating the mesh about the z-axis. The perceived need for excessive memory use with model `LZR_XBET_extrude` (1) was the reason for first developing the model `LZR_XBET_extrude` (0). To overcome the memory limitation all 3D data are packed using index indirection to track data only in illuminated cells. This can lead to large packing (compression) ratios since typically a small fraction of the mesh is illuminated by each beam and an even smaller fraction is illuminated by more than one beam concurrently. The packing scheme is dynamic: it may change during iteration and generally changes across cycles. It is used for fully 3D computations as well as the quasi-2D case. It is saved and reinstated when usng `LZR_XBET_istate`(1) and must be merged when using `LZR_replicates` > 1.

7 Hydra Implementation

The cross beam energy transfer model is enabled when the parameter

$$LZR\_XBET > 0.$$

Choosing the value (2) will cause the ion acoustic wave power to be locally deposited directly to the ion fluid. A value (1) will deposit this power to the electron fluid. Each group of beams acting as a unit for purposes of the `XBET` (Cross Beam Energy Transfer) coupling should have a unique `beam_gp`

modifier. The beam geometries, including beam wave-vectors, beam axes, spot sizes at best focus and frequencies for all beams in each beam group are averaged together for purposes of the coupling computation. Thus, one might choose to specify each beam in a quad separately (each with its own `superg3d` or `arbeampattern` card) conjoined with the same `beam_gp` modifier, or alternatively, an equivalent beam quad could be specified as a single beam.

The beam group frequency tuning is specified by the optional modifier `beam_frmult`, f_q. This modifier multiplies the frequency (harmonic) specifier h_m on the `laserbeam` card and is fixed for the entire run. Time/value control of the frequency for each `laserbeam` card, f_{tv}, is also provided separately with the `nfrtimes` card. The frequency for each beam group is thus the neodymium fundamental ω_0 multipled by the per `laserbeam` time dependent multiplier f_{tv} and the beam group multiplier f_q,

$$\omega_q = \omega_0 h_m f_{tv} f_q.$$

f_{tv} and f_q both default to 1 if they are not specified.

Hydra also provides the option to limit the coupling sums (53) to a pre-specified set of interacting beams by setting an additional modifier, `XBET_gp_id`, for each beam. This optional modifier to the `superg3d` or `arbeampattern` card is input similarly to the other optional modifiers `beam_id`, `beam_gp`, `beam_frmult` and `beam_label`. All beams with the same `XBET_gp_id` are included in the group sum, for that `XBET_gp_id` group. This can be useful, for example, to decouple oppositely pointed beams for purposes of the coupling calculation. Through this mechanism, beams whose interaction can be safely ignored may be excluded from each group sum (53), simplifying the calculation. The default, no beam having `XBET_gp_id` set, will cause *all* beams to be considered potentially interacting with each other.

To minimize the memory required to implement cross beam energy transfer, only the coefficients $\mathcal{C}_{qq'}^{n_e}$ and $\mathcal{C}_{qq'}^{Q}$ required for the sum over beam transfer rates in (61) for any given cell in any given cycle are computed and stored. The symmetries (79) and (80) are useful for compact storage of these coefficients which do not change as the ray tracing is iterated to determine the self consistent intensity. Only the values for $q > q'$ are needed.

For each interaction group with a number $N_g > 1$ beams interacting in the cell, the total number of coefficients to be stored for that cell for the cycle's iteration is $N_g(N_g - 1)/2$.

7.1 `LZR_replicates`

The LZR module in Hydra supports a procedure designed to optimize performance when scaling to large numbers of processes in a parallel computing environment. The parameter `LZR_replicates` can be chosen to replicate the mesh related date and distribute the replicates equally among the active processes. The rays are distributed as well generating a statistically equivalent representative ray ensemble in each replicate. One may consider each replicate in this mode as an independent realization unless `LZR_XBET > 1`. In that case the beam group intensities are accumulated across replicates with each iteration so that all rays are interacting as a single realization. The accumulation across replicates as well as the saving and reinstating of intensity state across cycles requires merging and saving the intensity packing scheme described in the previous section.

8 Stimulated Raman Scattering

Stimulated Raman scattering of laser light can be an important loss mechanism for hohlraums. Plasma wave (Langmuir) noise present in the illuminated hohlraum can interact with incident light to produce backscattered light which resonantly amplifies when the Langmuir frequency is the beat frequency of the pump (forward moving) and backscattered (reverse moving) waves. The interaction coupling calculation is similar to the cross beam coupling with the wave-vector directions of pump and scattered wave oppositely directed and the frequencies differing by the plasma frequency at resonance.

In reality there is a spectrum of backscattered waves observed corresponding to a range of plasma conditions where amplification is large enough to produce an observable signal. Furthermore, the backscattered waves refract differently from the pump beam due to their down-shifted frequencies. Both these facts complicate the modeling of SRS. Some simplifying approximations will serve to make the problem tractable without doing too much damage to the physics. (1) We ignore the difference of frequency between pump and scattered rays when computing the effect of refraction on ray trajectories; we pair pump and scattered rays along the same trajectories. (2) We choose a frequency and power for the scattered ray based on experimental data.

The basic result of these approximations will be to give a better estimate for the power deposition in the plasma both from inverse Bremsstrahlung

on pump and scattered waves and Landau damping on the driven Langmuir waves. The latter can be handled as a thermal power source or as a supra-thermal electron heat source.

The equations we use will be modifications of (30),(31)

$$\frac{\partial}{\partial \tau_1}\mathcal{I}_1 = \mathrm{p}_{12}\eta_1\alpha_2\mathcal{C}_{12}\,\mathcal{I}_2\mathcal{I}_1 \tag{83}$$

$$\frac{\partial}{\partial \tau_2}\mathcal{I}_2 = \mathrm{p}_{21}\eta_2\alpha_1\mathcal{C}_{21}\,\mathcal{I}_1\mathcal{I}_2 \tag{84}$$

in which the pump and Raman scattered wavevectors are oppositely directed and $\theta_{12} = \pi$ so

$$\mathrm{p}_{12} = \mathrm{p}_{21} = \frac{1}{4}(1 + \cos^2\theta_{12}) = \frac{1}{2}.$$

Letting $1 \to p$ represent the pump rays and $2 \to R$ represent the Raman reflected rays $\Delta k^2 = |\vec{k_p} - \vec{k_R}|^2 = (k_p + k_R)^2$ where $\vec{k}_{p,R}^{v}$ are the beam vacuum wave-numbers, $\vec{k_p} = \eta_p\vec{k}_p^v$, $\vec{k_R} = \eta_R\vec{k}_R^v$. Since for SRS $\chi_i \ll 1$ we adjust (21) as

$$F_{\chi_{(p,R)}} = \frac{\chi_{e,(p,R)}}{1 + \chi_{e,(p,R)}} \tag{85}$$

with $\chi_{e,(p,R)}(\Delta k, \Delta\omega, n_e, T_e)$ the electron susceptibilities for the pump and Raman rays. Adapting (25) we can write

$$\frac{\partial}{\partial \tau_p}\mathcal{I}_p = \frac{1}{2}\eta_p\alpha_R\mathcal{C}_{pR}\,\mathcal{I}_R\mathcal{I}_p \tag{86}$$

$$\frac{\partial}{\partial \tau_R}\mathcal{I}_R = \frac{1}{2}\eta_R\alpha_p\mathcal{C}_{Rp}\,\mathcal{I}_p\mathcal{I}_R \tag{87}$$

with

$$\mathcal{C}_{pR} = \frac{\Delta k^2}{4k_p}\frac{\Im(\chi_{e,p})}{1 + |\chi_{e,p}|^2 + 2\Re(\chi_{e,p})} \tag{88}$$

$$\mathcal{C}_{Rp} = \frac{\Delta k^2}{4k_R}\frac{\Im(\chi_{e,R})}{1 + |\chi_{e,R}|^2 + 2\Re(\chi_{e,R})}. \tag{89}$$

From (35), ignoring the Doppler shift relative to the frequency downscatter shift we have

$$\chi_{e,p} = \frac{-1}{2(\Delta k \lambda_{De})^2} Z'\left(\frac{\omega_R - \omega_p}{\sqrt{2}\omega_{pe}\Delta k \lambda_{De}}\right) \tag{90}$$

$$\equiv \alpha_{e,p} Z'(\beta_{e,p}) \tag{91}$$

$$\chi_{e,R} = \frac{-1}{2(\Delta k \lambda_{De})^2} Z'\left(\frac{\omega_p - \omega_R}{\sqrt{2}\omega_{pe}\Delta k \lambda_{De}}\right) \tag{92}$$

$$\equiv \alpha_{e,R} Z'(\beta_{e,R}) \tag{93}$$

$$\equiv \alpha_{e,p} Z'(-\beta_{e,p}) \tag{94}$$

The energy conservation for Raman scattering in our single frequency SRS model is

$$\frac{\sum_{r_p} \delta \overline{P}_{r_p}|_R}{\sum_{r_R} \delta \overline{P}_{r_R}|_p} = \frac{\alpha_R \eta_p \mathcal{C}_{pR}}{\alpha_p \eta_R \mathcal{C}_{Rp}} = -\frac{\alpha_R \eta_p k_R}{\alpha_p \eta_R k_p} = -\frac{\lambda_R^v}{\lambda_p^v} = -\frac{\omega_p}{\omega_R} = -\frac{1}{1 - \omega_{pe}^*/\omega_p} \tag{95}$$

where $\omega_{pe}^* = \omega_p - \omega_R$ is the electron plasma frequency at maximum gain (pre-specified by experiment). This can be viewed as corresponding to (56) for cross beam energy transfer: thus, as in (57)

$$\Delta P_{pR} \lambda_p^v = -\Delta P_{Rp} \lambda_R^v. \tag{96}$$

we can represent the power transferred to the Langmuir waves to be

$$\Delta P^{Lr} = \Delta P_{pR} + \Delta P_{Rp} = \Delta P_{pR}\left(\frac{\omega_{pe}^*}{\omega_p}\right). \tag{97}$$

With these definitions we can recast the electron susceptibilities as

$$\chi_{e,p} = \alpha_{e,p} Z'(-\sqrt{\alpha_{e,p}}\frac{\omega_{pe}^*}{\omega_{pe}}) \tag{98}$$

$$\chi_{e,R} = \alpha_{e,p} Z'(\sqrt{\alpha_{e,p}}\frac{\omega_{pe}^*}{\omega_{pe}}). \tag{99}$$

The crux of the SRS calculation will be to evaluate the Raman gain for each ray's cell traversal

$$\overline{\nu}_{SRSr_p} = \frac{1}{\tau_{r_p}} \int_0^{\tau_{r_p}} d\tau\, \nu_{SRSr_p}(\tau) \tag{100}$$

with

$$\nu_{SRSr_p} = \frac{1}{\delta \overline{P}_{r_p}} \frac{\partial}{\partial \tau_p} \delta \overline{P}_{r_p} = \frac{1}{2} \eta_p \alpha_R \mathcal{C}_{pR} \mathcal{I}_R$$

and, likewise

$$\nu_{SRSr_R} = \frac{1}{\delta \overline{P}_{r_R}} \frac{\partial}{\partial \tau_R} \delta \overline{P}_{r_R} = \frac{1}{2} \eta_R \alpha_p \mathcal{C}_{Rp} \mathcal{I}_p.$$

For purposes of computing the cell ray integral (101) we take the quantities $\eta_{p,R}$ and $\alpha_{p,R}$ to be constant in the cell and constant across intensity iterations. The quantities $\mathcal{I}_p$ and $\mathcal{I}_R$ are also treated as cell constants, but will generally vary across intensity iterations. The variation along rays which requires special care comes from the variation of $\alpha_{e,p} \sim n_e/T_e$ and $\omega_{pe} \sim \sqrt{n_e}$ in (100) which can produce resonance in $\mathcal{C}_{pR}$ and $\mathcal{C}_{Rp}$ near $\omega_{pe} \sim \omega_{pe}^*$. We use a rational function integration techique suggested by Ed Williams and detailed by D. Strozzi in Appendix D to compute the ray integral per cell traversal once per cycle; the result is stored with the intensity removed along the ray trajectory, then rescaling with each intensity iteration for computational efficiency. It is necessary to compute only one of $\overline{\nu}_{SRSr_p}, \overline{\nu}_{SRSr_R}$ per cell traversal using the scaling relation

$$\overline{\nu}_{SRSr_R} = -\overline{\nu}_{SRSr_p} \frac{\omega_R}{\omega_p} \frac{\mathcal{I}_p}{\mathcal{I}_R}.$$

A Normalizations

Michel's normalization for intensity, $\alpha^{-1}|\vec{a}|^2 = \mathcal{I}$ in units of W/cm^2, derives from the specification of the beam vector potential as

$$\vec{A} \sim \frac{m_e c^2}{2e}\vec{a}, \tag{101}$$

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} \tag{102}$$

$$\mathcal{I} = \frac{1}{2}\epsilon_0 v_g |\vec{E}|^2 \tag{103}$$

so that α^{-1} evaluates to

$$\alpha^{-1} = k^v \omega \frac{\epsilon_0}{2}(\frac{(m_e c^2)}{e})^2 \frac{W}{cm^2} \tag{104}$$

$$\alpha^{-1} = 1.2185 \times 10^{18}\frac{\lambda_0^2}{\lambda^2}\frac{W}{cm^2} \tag{105}$$

$$\alpha^{-1} = \frac{\lambda_0^2}{\lambda^2} \tag{106}$$

$$* \frac{1}{2}8.854 \times 10^{-12}\frac{C^2}{Jm} \tag{107}$$

$$* (\frac{.511 \times 10^6 eV}{1.602176 \times 10^{-19}C})^2 \tag{108}$$

$$* 3 \times 10^{10}\frac{cm}{sec} \tag{109}$$

$$* (\frac{2\pi}{1.06 \times 10^{-4}cm})^2 \tag{110}$$

$$* (\frac{J}{6.24151 \times 10^{18}eV})^2 \tag{111}$$

$$* \frac{1m}{100cm} \tag{112}$$

$$\tag{113}$$

The λ values are the vacuum wavelengths with λ_0 the $1.06\mu m$ value.

B Dawson's Fuction

```
#include <math.h>

#define nmax 6
#define h 0.4
#define a1 (2.0/3.0)
#define a2 0.4
#define a3 (2.0/7.0)
double DawsonF(double x)
// Returns Dawson's integral
// D(x) = ∫₀ˣ e^{t²}dt/e^{x²}  for any real x
{
  int i,n0;
  double d1,d2,e1,e2,sum,x2,xp,xx,ans;
  static double c[nmax+1];
  // Flag is 0 if we need to initialize, else 1.
  static int init = 0;
  if (init == 0) {
    init=1;
    for (i=1;i<=nmax;i++) {
      double q = h*(2.0*i-1.0);
      c[i]=exp(-q*q);
    }
  }
  if (fabs(x) < 0.2) { // Use series expansion.
    x2=x*x;
    ans=x*(1.0-a1*x2*(1.0-a2*x2*(1.0-a3*x2)));
  } else { // Use sampling theorem representation.
    xx=fabs(x);
    n0=2*(int)(0.5*xx/h+0.5);
    xp=xx-n0*h;
    e1=exp(2.0*xp*h);
    e2=e1*e1;
    d1=n0+1;
    d2=d1-2.0;
    sum=0.0;
    for (i=1;i<=nmax;i++,d1+=2.0,d2-=2.0,e1*=e2)
      sum += c[i]*(e1/d1+1.0/(d2*e1));
    // Constant is 1 = 1/√π
    ans=0.5641895835*SIGN(exp(-xp*xp),x)*sum;
  }
  return ans;
}
```

C `LZR_XBET_extrude` (0)

For the 2D XBET computation chosen by `LZR_XBET_extrude` (0), the beam group illuminated volume, its envelope, is modeled by an elliptical cylinder whose axis is aligned with the beam group direction. The crossection of the cylinder is given at the best focus by the best focus spot sizes. Along the beam axis, this crossection is expanded linearly as specified by the optical aperture number of the lens, F, computed from the beam group divergence angle, a_2.

Beam group coincidence is approximated as the fractional overlap of the two beam groups in the extrusion ring of the traversal cell around the z-axis. This fraction, $\epsilon_{qq'}(r, z)$, is computed once per cycle for each cell where rays from more than one beam group contibute to the intensity. For rings entirely contained within a beam group envelope, all rays associated with that beam group contibute to the intensity and the fractional overlap is 1.

Let the beam group pointing direction, $\hat{q}$, launch point, $\hat{p}$, and best focus, $\vec{f}$, in the beam definition frame be

$$\hat{z}_b = \hat{q} = (\hat{q}_x, \hat{q}_y, \hat{q}_z) \equiv R_b \hat{z} \tag{114}$$

$$\hat{x}_b = R_b \hat{x} \tag{115}$$

$$\hat{y}_b = R_b \hat{y} \tag{116}$$

$$\vec{p} = (p_x, p_g y, p_z) \tag{117}$$

$$\vec{f} = (f_x, f_y, f_z). \tag{118}$$

Rotating the beam definition frame in azimuth R_ϕ gives

$$\vec{p}' = R_\phi \vec{p} \tag{119}$$

$$\vec{f}' = R_\phi \vec{f} \tag{120}$$

$$\hat{x}_b' = R_\phi R_b \hat{x} \tag{121}$$

$$\hat{y}_b' = R_\phi R_b \hat{y} \tag{122}$$

$$\hat{z}_b' = \hat{q}' = R_\phi \hat{q} = (\vec{p}' - \vec{f}')/|(\vec{p}' - \vec{f}')| \tag{123}$$

For a given cell center at $\vec{x_c} = (r_c, z_c)$, for each beam group, the z-distance from the z-plane at z_c to best focus is

$$\delta z_f = z_c - f_z';$$

the radius of the beam axis intersection with the z-plane is then

$$r_{bc} = \delta z_f \tan(\theta) + r'_f.$$

The angle θ here is the beam group polar inclination, $\hat{z}'_b \cdot \hat{z}$ and $r'_f = \sqrt{f'^2_x + f'^2_y}$ is the radius (distance from $the z$-axis) of the center of the focal spot. Given r_{bc}, r_c and the envelope for each beam group, two beam groups (qq') at a time, the overlap $\epsilon_{qq'}(r_c, z_c)$ can be computed for any cell.

One last consideration is the effective coupling strength. For the quasi-2D calculation, ray powers are reduced by the relative size of the wedge, $\Delta\phi_w/2\pi$. That is to say, the beam power is averaged over the entire cylinder. To correct for this reduction in the coupling calculation, we amplify the coupling strength by the inverse of the actual reduction in beam power due to this averaging, namely $\mathcal{A}_q = 2\pi/\Delta\phi_q$, where $\Delta\phi_q$ is the azimuthal extent of the the beam group envelope along the extrusion ring of the traversal cell around the z-axis. We set $\mathcal{A}_q = 1$ if the entire ring is contained within the beam envelope.

With these considerations, the 3D gain coefficients given in (73) are modified to

$$\mathcal{C}^{n_e}_{qq'} \rightarrow \mathcal{C}^{n_e \, *}_{qq'} = \mathcal{C}^{n_e}_{qq'} \sqrt{(\mathcal{A}_q \mathcal{A}_{q'} \epsilon_{qq'} \epsilon_{q'q})} \tag{124}$$

and

$$\mathcal{C}^{Q}_{qq'} \rightarrow \mathcal{C}^{Q \, *}_{qq'} = \mathcal{C}^{Q}_{qq'} \sqrt{(\mathcal{A}_{q'} \epsilon_{q'q} / \mathcal{A}_q \epsilon_{qq'})}. \tag{125}$$

These changes lead directly to the 2D version of the saturation corrected gain coefficients

$$\nu_{qq'} = \left| \frac{\delta\hat{n}_e}{n_e} \right|_{q,q'} Q_{q,q'} = \mathcal{C}^{n_e \, *}_{qq'} \sqrt{\mathcal{I}_q \mathcal{I}_{q'}} \, \mathcal{C}^{Q \, *}_{qq'} \sqrt{\mathcal{I}_{q'} / \mathcal{I}_q} \tag{126}$$

$$= \mathcal{C}^{n_e \, *}_{qq'} \mathcal{C}^{Q \, *}_{qq'} \mathcal{I}_{q'} \qquad\qquad \Longleftrightarrow \mathcal{C}^{n_e \, *}_{qq'} \sqrt{\mathcal{I}_q \mathcal{I}_{q'}} < S_t \tag{127}$$

$$= S_t \mathcal{C}^{Q \, *}_{qq'} \sqrt{\mathcal{I}_{q'} / \mathcal{I}_q} \qquad\qquad \Longleftrightarrow \mathcal{C}^{n_e \, *}_{qq'} \sqrt{\mathcal{I}_q \mathcal{I}_{q'}} >= S_t. \tag{128}$$

D   Strozzi's Function

Referring to the current document, Eq. (87) governs the SRS reflected light intensity:

$$\frac{1}{I_R}\frac{\partial I_R}{\partial \tau_R} = \frac{1}{2}\eta_R\alpha_p C_{Rp}I_p \tag{129}$$

We take I_p to be given, i.e. pump depletion is not done self-consistently. We drop the subscript R on τ_R. The RHS is then a known function of τ, and the solution from $\tau = \tau_0$ to τ_1 is

$$I_R(\tau_1) = I_R(\tau_0)\exp[G] \qquad G \equiv \int_{\tau_0}^{\tau_1}\frac{1}{2}\eta_R\alpha_p C_{Rp}I_p d\tau \tag{130}$$

G is the SRS intensity gain exponent. To connect with the previous notation, we define $G = \nu_{SRS}\Delta\tau$ with $\Delta\tau \equiv \tau_1 - \tau_0$. If we assume I_p is constant with respect to τ, we find

$$G = I_p g \qquad g \equiv \int_{\tau_0}^{\tau_1}\Gamma d\tau \qquad \Gamma \equiv \frac{1}{2}\eta_R\alpha_p C_{Rp} \tag{131}$$

g is independent of intensity and is just a function of plasma conditions. It does not need to be re-computed with each intensity iteration. Plugging in definitions, we obtain

$$\Gamma = \Gamma_0\Im\frac{\chi}{1+\chi} = \Gamma_0\frac{\Im\chi}{|1+\chi|^2} \tag{132}$$

$$\Gamma_0 \equiv \left(\frac{1}{8}k_R^v\alpha_p\right)\left(\frac{k_p^v}{k_R^v}\eta_p + \eta_R\right)^2 \tag{133}$$

$\chi = \chi_{e,R}$. The first parenthesis in Γ_0 is independent of τ and contains all the units. The second parenthesis, and the final fraction in Γ, are both unitless and depend on τ.

We can use the *ratint* approach to rational function integrals developed by Ed Williams to find g. For now, we treat Γ_0 as constant, and assume we know χ at τ_0 and τ_1 and that it varies linearly between these two points. We use the second form of Γ and explicitly treat everything as real. One can also use the first form and get an elegant answer with complex logs - assuming you handle all the branch points right and have a complex log function available. It is also convenient to use $\epsilon = 1 + \chi$ instead of χ. We have

$$g = \Gamma_0 g_\epsilon \qquad g_\epsilon \equiv \int_{\tau_0}^{\tau_1} \frac{\epsilon_i}{\epsilon_r^2 + \epsilon_i^2} d\tau \qquad (134)$$

where $\epsilon_r = \Re\epsilon$ and $\epsilon_i = \Im\epsilon$.

We wish to find g_ϵ given four real numbers: $\epsilon_{r0}, \epsilon_{r1}, \epsilon_{i0}, \epsilon_{i1}$. Put

$$\epsilon_r = \epsilon_{r0} + u\Delta_r, \qquad u \equiv \frac{\tau - \tau_0}{\Delta\tau}, \qquad \Delta_r \equiv \epsilon_{r1} - \epsilon_{r0} \qquad (135)$$

and similarly for ϵ_i. Change variables to u to obtain

$$\frac{g_\epsilon}{\Delta\tau} = \int_0^1 \frac{\epsilon_{i0} + u\Delta_i}{(\epsilon_{r0} + u\Delta_r)^2 + (\epsilon_{i0} + u\Delta_i)^2} du \qquad (136)$$

Re-arranging,

$$\frac{g_\epsilon}{\Delta\tau} = \int_0^1 \frac{\epsilon_{i0} + u\Delta_i}{D_0 + D_1 u + D_2 u^2} du \qquad (137)$$

with

$$D_0 \equiv \epsilon_{r0}^2 + \epsilon_{i0}^2 \qquad D_1 \equiv 2\left[\epsilon_{r0}\Delta_r + \epsilon_{i0}\Delta_i\right] \qquad D_2 \equiv \Delta_r^2 + \Delta_i^2 \qquad (138)$$

Note that $D_2 \geq 0$. If $D_2 = 0$, that means both Δ_r and Δ_i are zero, $D_1 = 0$ as well, and $\epsilon_0 = \epsilon_1$. Our linear interpolation within the zone means ϵ is constant. We then have

$$\frac{g_\epsilon}{\Delta\tau} = \frac{\epsilon_{i0}}{\epsilon_{r0}^2 + \epsilon_{i0}^2} \qquad D_2 = 0 \qquad (139)$$

We now treat the case $D_2 > 0$. Multiply both sides by D_2 to find

$$\frac{D_2 g_\epsilon}{\Delta\tau} = \int_0^1 \frac{\epsilon_{i0} + u\Delta_i}{d_0 + d_1 u + u^2} du \qquad d_{0,1} \equiv D_{0,1}/D_2 \qquad (140)$$

Change variables to $v = u + d_1/2$:

$$\frac{D_2 g_\epsilon}{\Delta\tau} = (\epsilon_{i0} - \Delta_i d_1/2)J_0 + \Delta_i J_1 \qquad J_j \equiv \int_{v_0}^{v_1} \frac{v^j}{v^2 + d_3^2} dv \qquad j = 0, 1 \quad (141)$$

with $v_0 = d_1/2$, $v_1 = 1 + v_0$, and $d_3^2 \equiv d_0 - d_1^2/4$. Some algebra gives

$$d_3 = \frac{\Delta_i \epsilon_{r0} - \Delta_r \epsilon_{i0}}{D_2} \tag{142}$$

so that d_3^2 is indeed non-negative. d_3^2 as defined can have either sign, and our answers correctly handle both cases, that is, they are really functions just of $|d_3|$.

For J_0 we have

$$J_0 = \frac{1}{d_3} \left[\arctan \frac{v_1}{d_3} - \arctan \frac{v_0}{d_3} \right] \tag{143}$$

Since arctan is odd, J_0 has the same form if we change d_3 to $-d_3$. Also,

$$\frac{v_j}{d_3} = \frac{\Delta_r \epsilon_{rj} + \Delta_i \epsilon_{ij}}{\Delta_i \epsilon_{r0} - \Delta_r \epsilon_{i0}} \qquad j = 0, 1 \tag{144}$$

Note the j subscript in the numerator but 0 subscript in the denominator. J_1 gives the simple result

$$J_1 = \ln \left| \frac{\epsilon_1}{\epsilon_0} \right| = \ln \frac{\epsilon_{r1}^2 + \epsilon_{i1}^2}{\epsilon_{r0}^2 + \epsilon_{i0}^2} \tag{145}$$

Putting it all together, for $D_2 > 0$,

$$\frac{D_2 g_\epsilon}{\Delta \tau} = -\Delta_r \left[\arctan \frac{\Delta_r \epsilon_{r1} + \Delta_i \epsilon_{i1}}{\Delta_i \epsilon_{r0} - \Delta_r \epsilon_{i0}} - \arctan \frac{\Delta_r \epsilon_{r0} + \Delta_i \epsilon_{i0}}{\Delta_i \epsilon_{r0} - \Delta_r \epsilon_{i0}} \right] + \Delta_i \frac{1}{2} \ln \left[\frac{\epsilon_{r1}^2 + \epsilon_{i1}^2}{\epsilon_{r0}^2 + \epsilon_{i0}^2} \right] \tag{146}$$